We present how a neutrino condensate and small neutrino masses emerge from a topological formulation of gravitational anomaly. In a recent paper it was shown that a gravitational $\theta$-term leads to the emergence of a new bound neutrino state analogous to the $\eta'$ meson of QCD. Building on this result, we compute the consequent formation of a neutrino vacuum condensate, which effectively generates small neutrino masses. We investigate the numerous phenomenological consequences of our neutrino mass generation model. The cosmological neutrino mass bound vanishes since we predict the neutrinos to be massless until the phase transition in the late universe, $T \sim \text{meV}$. Coherent radiation of new light particles in the neutrino sector can be detected in prospective precision experiments. The deviations from an equal flavor rate due to enhanced neutrino decays in extraterrestrial neutrino fluxes can be observed in future IceCube data. The current cosmological neutrino background only consists of the lightest neutrinos, which, due to enhanced neutrino-neutrino interactions, either bind up, form a superfluid, or completely annihilate into massless bosons. The strongly coupled relic neutrinos could provide a contribution to cold dark matter in the late universe, together with the new proposed particles and topological defects, which may have formed during neutrino condensation. The enhanced interactions may also solve the puzzle of the reactor antineutrino anomaly and could be a source of relic neutrino clustering in our galaxy, which possibly makes the overdense cosmic neutrino background detectable in the KATRIN experiment. The neutrino condensate provides a mass for the hypothetical $B - L$ gauge boson, leading to a gravity-competing force detectable in short-distance measurements. The prospective measurement of the polarization intensities of gravitational waves can falsify our neutrino mass generation model.